Approximation of truncated Beta operator of Max-product kind

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**Abstract.**

In this paper, we studied the Shepard nonlinear operator of max-product Beta operators. Also, we estimate the order of uniform approximation for the function \( f \in C[0,1] \) to study the truncated of Beta operator of max-product. We proves that the order of uniform approximation in the general case of this kind of the approximation is \( \omega_1(f,.) \) cannot be improved.

**Key words and phrases**, Nonlinear truncated Beta operator of max-product kind, degree of approximation, shape preserving properties.

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1. Introduction

Recently, many studies deal with the max-product and the Shepard nonlinear operator for many sequences of operators were starting the open problem 5.5.4, pp.324-326 in [4]. This branch began with Bede et al. in 2006 and 2008 when they studied the Shepard nonlinear operators of max-product kinds. After that, in 2010, Barnabas et al. introduced the max-product type operators for Favard-Szász-Mirakjan. In 2011 Barnabas et al. studied the nonlinear max-product Baskakov operators.

In this paper, we study the Shepard nonlinear operator of max-product Beta operators of max-product. We define these operators as follows:

\[ R_{n,M}(f)(x) = \frac{\sum_{k=0}^{\infty} \beta_{n,k} \left( \frac{k}{n} \right) f \left( \frac{k}{n} \right)}{\sum_{k=0}^{\infty} \beta_{n,k} \left( \frac{n}{n} \right)}, n \in N = \{1,2,...\} \]

Note that the notation \( \vee \) denotes the maximum, where

\[ \beta_{n,k}(x) = \frac{(n+k)!}{k!(n-1)!} x^k (1+k)^{-n-k-1}, \quad n \in N, x \in [0,1]. \]

Also, we introduce the order of uniform approximation for the function \( f \in C[0,1] \) to study the truncated of Beta operators of max-product, which is defined as follows:

\[ B_{n,M}(f)(x) = \frac{\sum_{k=0}^{n} \beta_{n,k} \left( \frac{k}{n} \right) f \left( \frac{k}{n} \right)}{\sum_{k=0}^{n} \beta_{n,k} \left( \frac{n}{n} \right)} x \in [0,1] \text{ and } n \in N, n \geq 1. \]

Then in general case of max-product of the approximation we prove that the order approximation is \( \omega_1(f,\cdot) \) cannot be improved.

Let \( \mathbb{R}^+ \) denotes the set of all positive real numbers, the operation \( \vee \) is maximum and . is the product. Hence, we have \((\mathbb{R}^+; \vee, .)\) is a semiring structure and said to be a max-product algebra(by the operation on semiring structure and semiring properties)[1].

Let \( I \) be a closed (bounded or unbounded) interval and \( CB(I) = \{ f: I \rightarrow \mathbb{R}^+; f \text{ continuous and bounded on } I \} \). Then we obtain the general form \( L_n: CB(I) \rightarrow CB_+(I) \) and \( (L_n \text{ said to be the discrete max-product type approximation operator}) \)[1].

\[ L_n(f)(x) = \bigvee_{k=0}^{n} k_n(x,x_i)f(x_i) \]

and

\[ L_n(f)(x) = \bigvee_{k=0}^{\infty} k_n(x,x_i)f(x_i), \]

where \( n \in N, f, k_n(\cdot, k_i) \in CB_+(I) \) and \( x_i \in I \) for any \( i \).

These operators are nonlinear, positive operators and further that satisfy a pseudo linearity condition,

\[ L_n(\delta \cdot f \vee \beta \cdot g) = \delta \cdot L_n(f)(x) \vee \beta \cdot L_n(g)(x), \]

for all \( \delta, \beta \in \mathbb{R}^+, f, g: I \rightarrow \mathbb{R} [1]. \)

First, we give the following Lemma which shows some properties of the operators \( L_n \).

**Lemma 1.1.**[1]

Let \( I \in \mathbb{R} \) be a bounded or unbounded interval,
\( CB_+(I) = \{ f: I \to \mathbb{R}, f \text{ continuous and bounded on } I \} \),
and \( L_n: CB_+(I) \to CB_+(I), n \in N \), be a sequence of operators satisfying the following properties:
(i) If \( f, g \in CB_+(I) \) satisfy \( f \leq g \) then \( L_n(f) \leq L_n(g) \) for all \( n \in N \);
(ii) \( L_n(f + g) \leq L_n(f) + L_n(g) \) for all \( f, g \in CB_+(I) \).
Then for all \( f, g \in CB_+(I), n \in N \) and \( x \in I \) we have
\[
\left| L_n(f)(x) - L_n(g)(x) \right| \leq L_n(\| f - g \|)(x).
\]

**Remark 2.1.** [1]

1. One can check that the truncated Beta max-product operator satisfy the condition of Lemma 2.1, (i), (ii), and satisfies the stronger condition is
\[
L_n(f \lor g)(x) = L_n(f)(x) \lor L_n(g)(x), \quad f, g \in CB_+(I).
\]
Indeed, we take in the above equality \( f \leq g \), \( f, g \in CB_+(I) \). It easily follows:
\[
L_n(f)(x) \leq L_n(g)(x).
\]
2. In addition, we see that the truncated Beta max-product operator is positive homogenous, i.e.
\[
L_n(\rho f) = \rho L_n(f), \quad \forall \rho \geq 0.
\]
3. Since in the main results take \( I = [0,1] \) the following two Corollaries are stated here just that the interval \( I \) is bounded and not unbounded.

**Corollary 1.1.** [1]

Let \( I \) be bounded or unbounded interval, \( L_n: CB_+(I) \to CB_+(I), \ n \in N \), be a sequence of operators satisfying the condition (i), (ii) in Lemma 2.1, and in addition begin positively homogenous. Then for all \( f \in CB_+(I), n \in N \), and \( x \in I \) we have:
\[
\left| f(x) - L_n(f)(x) \right| \leq \frac{1}{\delta} L_n(\psi_x)(x) + L_n(e_0)(x)\omega_1(f; \delta)_I + \left| f(x) \right| \cdot \left| L_n(e_0)(x) - 1 \right|,
\]
where \( \omega_1(f; \delta)_I = \max\{ \left| f(x) - f(y) \right|; x, y \in I, \left| x - y \right| \leq \delta \} \).
An immediate consequence of Corollary 1.1. is the following:

**Corollary 1.2.** [1]

Suppose that in addition to the condition in the Corollary 1.1, the sequence \( (L_n)_n \) satisfies \( L_n(e_0) = e_0 \), for all \( n \in N \). Then for all \( f \in CB_+(I), n \in N \) and \( x \in I \) we have
\[
\left| f(x) - L_n(f)(x) \right| \leq [1 + \frac{1}{\delta} L_n(\psi_x)(x)]\omega_1(f; \delta)_I.
\]

**2. Auxiliary Result**

We state some auxiliary results which are help as in proving the main results.

**Remark 2.2.**

From the bounded interval which is \( I = [0,1] \), note that the consideration, is that \( B_{nM}(f)(x) \) satisfy the conditions in Lemma 1.1., Corollary 1.1. and Corollary 1.2.

**Lemma 2.1.**

The operator \( B_{nM}(f)(x) \) is positive, bounded, continuous on \( [0,1] \), for any arbitrary function \( f: [0,1] \to \mathbb{R}^+ \) and satisfies \( B_{nM}(f)(0) = f(0) \) for all \( n \in N \).

**Proof.**

From, the operator \( B_{nM}(f)(x) \) coincides with \( f(x) \) at \( x = 0 \) directly follows from the consideration which is \( \beta_n(x) > 0 \) for all \( x \in (0,1], n \in N, k \in \{0,1,2,\ldots, n\} \), it follows that the denominator \( \sum_{k=0}^{n} \beta_n(x) > 0 \) for all \( x \in (0,1] \), and \( n \in N \).
But the numerator is a maximum of finite number of continuous function on \([0,1]\), so it is a continuous function on \([0,1]\) and this implies that \(B_{n,M}(f)(x)\) is continuous on \((0,1)\).

To prove that the continuity of \(B_{n,M}(f)(x)\) at \(x = 0\), we observe that \(\beta_{n,k}(x) = 0\), for all \(k \in \{0,1,2,\ldots,n\}\) and \(\beta_{n,k}(x) = 1\) for \(k = 0\) which implies that \(\bigvee_{k=0}^{n} \beta_{n,k}(x) = 1\) in the case of \(x = 0\).

The operator \(B_{n,M}(f)(x)\) coincides with \(f(x)\) at \(x = 0\) directly follows from the above consideration.

Which proves the Lemma.

\(\Box\)

**Remark 2.3.**

In view of Lemma 2.1, we have \(B_{n,M}(f)(0) = f(0)\) for all \(n\), through it, follow that in the notations, proofs and statement of all approximation results, in fact we always may suppose that \(x > 0\). For each \(n \in \mathbb{N}\), \(n \geq 1, k \in \{0,1,2,\ldots,n\}\), \(j \in \{0,1,2,\ldots,n-1\}\) and \(x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]\), let us denote \(M_{k,n,j}(x) = m_{k,n,j}(x) \left\lfloor \frac{k}{n} - x \right\rfloor\),

where \(m_{k,n,j}(x) = \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)}\), for all \(x \in (0,1]\) and \(k \in \{1,2,\ldots,n\}\). Clearly that \(m_{0,n,0}(0) = 1\), \(\forall x \in (0,1]\), by the same way, we have: \(m_{k,n,0}(0) = 0\). For the Lemma 2.1, if \(k > j\) then \(M_{k,n,j}(x) = m_{k,n,j}(x) \left(\frac{k}{n} - x\right)\) and if \(k \leq j\) then \(M_{k,n,j}(x) = m_{k,n,j}(x) \left(x - \frac{k}{n}\right)\).

To prove the mains results, we need to introduce some auxiliary results, such as:

**Lemma 2.2.**

Let \(n \in \mathbb{N}\). For all \(k \in \{0,1,2,\ldots,n\}, j \in \{0,1,2,\ldots,n-1\}\) and \(x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]\) we have

\[m_{k,n,j}(x) \leq 1.\]

**Proof.**

First let \(x = 0\) we have \(j = 0\) it is clearly \(m_{0,n,0}(x) = 1\), by the same way we have: \(m_{k,n,0}(0) = 0\) for all \(k \in \{0,1,2,\ldots,n\}\), Suppose that \(x > 0\) then, it is clearly that \(m_{k,n,j}(x) > 0\), there is two cases: 1) \(k \geq j\) and 2) \(k \leq j\).

**Case 1.** Since the function \(h(x) = \frac{1 + x}{x}\), it is nonincreasing on \(\left[\frac{j}{n}, \frac{j+1}{n}\right]\) and (when \(j = 0\) then

\[m_{k,n,j}(x) \geq m_{k,n,j}(0) \leq (n + k)(n + k + 1) \frac{1 + \frac{j}{n} + 1}{x} = \frac{(n + 1)(n + j + 1)}{(n + k + 1)(j + 1)} \geq 1.\]

which implies that

\[m_{j,n,j}(x) \geq m_{j+1,n,j}(x) \geq m_{j+2,n,j}(x) \geq \cdots \geq m_{n,n,j}(x).\]

**Case 2.** since

\[
\frac{M_{k,n,j}(x)}{M_{k-1,n,j}(x)} = \frac{(n + k)!x^{k}(1 + x)^{-n-k-1}}{k!(n - 1)!} \cdot \frac{(k - 1)! (n - 1)!}{(n + k - 1)!x^{k-1}(1 + x)^{-n-k}} \cdot \frac{k + 1}{k} \cdot \frac{x}{1 + x}.
\]
Since $m_{j,n,j}(x) = 1$ and this proves the Lemma.

**Lemma 2.3.**

Let $x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right]$ and $n \in N$.

(i) If $k \in \{j + 3, j + 4, ..., n - 1\}$ is such that $k - \sqrt{2}(k + 1) \geq j$, then $M_{k,n,j}(x) \geq M_{k+1,n,j}(x)$.

(ii) If $k \in \{1, 2, ..., j - 1\}$ is such that $j - \sqrt{2j} \geq k$, then $M_{k,n,j}(x) \geq M_{k-1,n,j}(x)$.

**Proof.**

(i) We note that

$$M_{k,n,j}(x) = \frac{m_{k,n,j}(x)}{M_{k+1,n,j}(x)} = \frac{m_{k,n,j}(x) \left( \frac{k}{n} - x \right)}{m_{k+1,n,j}(x) \left( \frac{k + 1}{n} - x \right)}$$

$$= \frac{(n + k)! x^k (1 + x)^{-n-k-1}}{k! (n - 1)!} \cdot \frac{(k + 1)! (n - 1)!}{(n + k + 1)! x^{k+1} (1 + x)^{-n-k-2}} \cdot \frac{k}{n} - x$$

$$= \frac{k + 1}{n + k + 1} \cdot \frac{1 + x}{x} \cdot \frac{k}{n} - x \cdot \frac{k + 1}{n} - x$$

is nondecreasing, it follows that

$$g(x) \leq g \left( \frac{j + 1}{n} \right) = \frac{1 + j + 1}{j + 1} \cdot \frac{k + 1}{n} - j + 1 \cdot \frac{k - j - 1}{n} \cdot \frac{n + j + 1}{k + 1} \cdot \frac{k - j - 1}{k - j}$$

for all $x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right]$. Then $M_{k+1,n,j}(x) \leq M_{k,n,j}(x) \frac{k + 1}{n} \cdot \frac{n + j + 1}{k + 1} \cdot \frac{k - j - 1}{k - j}$

Since the condition $k - \sqrt{2(k + 1)} \geq j$, and using calculation

We obtain $(k + 1)(n + j + 1)(k - j - 1) - (n + k + 1)(j + 1)(k - j)$

$$= n[(k - j)^2 - (k + 1)] - (k + 1)(j + 1).$$

(ii) we note that

$$M_{k,n,j}(x) = \frac{(n + k)! x^k (1 + x)^{-n-k-1}}{k! (n - 1)!} \cdot \frac{(k - 1)! (n - 1)!}{(n + k - 1)! x^{k-1} (1 + x)^{-n-k}} \cdot \frac{x}{n} - \frac{k}{n}$$

$$= \frac{n + k}{k} \cdot \frac{x}{1 + x} \cdot \frac{x}{x - \frac{k - 1}{n}}$$

since the function $h(x) = \frac{x}{1 + x} \cdot \frac{x - \frac{k}{n}}{x - \frac{k - 1}{n}}$ is nondecreasing,

it follows that $h\left( \frac{j}{n} \right) = \frac{j}{n + j} \cdot \frac{j - k}{j - k + 1}$
for all $x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right]$. Then $M_{k,n,j}(x) \geq M_{k-1,n,j}(x) \frac{n+k}{n+j} \cdot \frac{j-k}{j-k+1}$.

Since the condition $j - \sqrt{2j} \geq k$, and using calculation we obtain

$(n + k)j(j-k) - k(n+j)(j-k+1) = n[(j-k)^2 - k] - kj \geq 0$ which prove Lemma □

**Lemma 2.4.**

Let $n \in \mathbb{N}$, we have $V_{k=0}^n \beta_{n,k}(x) = \beta_{n,j}(x)$, for all $x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right], j \in \{0,1,2,\ldots,n-1\}$.

**Proof.**

First we show that for fixed $n \in \mathbb{N}$ and $0 \leq k \leq k + 1 \leq n$ we have

$0 \leq \beta_{n,k+1}(x) \leq \beta_{n,k}(x)$ if and only if $x \in \left[ 0, \frac{k+1}{n} \right]$.

$0 \leq \frac{(n + k + 1)!}{(k+1)!(n-1)!} x^{k+1}(1+x)^{-n-k-2} \leq \frac{(n + k)!}{k!(n-1)!} x^k(1+x)^{-n-k-1}$.

Which after simple calculation is obviously equivalent to $0 \leq x \leq \frac{k+1}{n}$, now, by taking $k = 0,1,2,\ldots,n-1$, in the inequality just prove above, we obtain

$\beta_{n,1}(x) \leq \beta_{n,0}(x)$ if and only if $x \in \left[ 0, \frac{1}{n} \right]$,

$\beta_{n,2}(x) \leq \beta_{n,1}(x)$ if and only if $x \in \left[ 0, \frac{2}{n} \right]$,

$\beta_{n,3}(x) \leq \beta_{n,2}(x)$ if and only if $x \in \left[ 0, \frac{3}{n} \right]$,

so on, $\beta_{n,k+1}(x) \leq \beta_{n,k}(x)$ if and only if $x \in \left[ 0, \frac{k+1}{n} \right]$ and so on until finally,

$\beta_{n,n-1}(x) \leq \beta_{n,n-2}(x)$ if and only if $x \in [0,1]$ and

$\beta_{n,n}(x) \leq \beta_{n,n-1}(x)$ if and only if $x \in [0,1]$.

From all these inequalities, reasoning by recurrence we easily obtain,

- if $x \in \left[ 0, \frac{1}{n} \right]$ then $\beta_{n,k}(x) \leq \beta_{n,0}(x)$ for all $k = 0,1,2,\ldots,n$,
- if $x \in \left[ \frac{1}{n}, \frac{2}{n} \right]$ then $\beta_{n,k}(x) \leq \beta_{n,1}(x)$ for all $k = 0,1,2,\ldots,n$,
- if $x \in \left[ \frac{2}{n}, \frac{3}{n} \right]$ then $\beta_{n,k}(x) \leq \beta_{n,2}(x)$ for all $k = 0,1,2,\ldots,n$,

and in the case general then if $x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right]$ then $\beta_{n,k}(x) \leq \beta_{n,j}(x)$ for all $k = 0,1,2,\ldots,n$.

From the application above we can write the equation as follows:

$V_{k=0}^n \beta_{n,k}(x) = \max \{ V_{k=0}^{j-1} \beta_{n,k}(x), V_{k=j}^n \beta_{n,k}(x) \}$, which prove of Lemma □.

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Suppose that, for any $n \in \mathbb{N}, k \in \{0,1,2,\ldots,n\}$ and $j \in \{0,1,2,\ldots,n-1\}$, let us define the functions $f_{k,n,j} : \left[ \frac{j}{n}, \frac{j+1}{n} \right] \rightarrow \mathbb{R}$,

$f_{k,n,j}(x) = m_{k,n,j}(x)f(\frac{k}{n}) = \beta_{n,k}(x)f(\frac{k}{n}) = \frac{(n+k)!}{(n+j)!} \cdot \frac{j!}{k!} \left( \frac{x}{1+x} \right)^{k-j} f(\frac{k}{n})$.

For any $j \in \{0,1,2,\ldots,n-1\}$ and $x \in \left[ \frac{i}{n}, \frac{i+1}{n} \right]$ we can write $B_{n,M}(f)(x) = V_{k=0}^n f_{k,n,j}(x)$. 
4. Approximation Results

Statement not clear, if \( B_{n,M}(f)(x) \) represents the truncated Beta operator of max-product kind, then the first main result of this section is the following.

**Lemma 2.5.**

Let \( f : [0,1] \rightarrow [0, \infty) \) be such that \( B_{n,m}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\} \) for all \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \) and \( n \in N \). Then \( |B_{n,m}(f)(x) - f(x)| \leq \omega_1 \left( f; \frac{1}{n} \right) \) for all \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \).

**Proof.**

We take two cases: **Case 1).** Let \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \) is fixed such that \( B_{n,m}(f)(x) = f_{k,n,j}(x) \).

By simple calculation we have \( 0 \leq x - \frac{j}{n} \leq \frac{j+1}{n} - \frac{j}{n} = \frac{1}{n} \leq \frac{1}{n}, \) and \( f_{j,n,j}(x) = f \left( \frac{j}{n} \right) \), it follows that \( |B_{n,m}(f)(x) - f(x)| \leq \omega_1 \left( f; \frac{1}{n} \right) \).

**Case 2).** Let \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \) be such that \( B_{n,m}(f)(x) = f_{j+1,n,j}(x) \). We have two subcases:

a) \( B_{n,m}(f)(x) \leq f(x) \), when obviously, \( f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x) \) and we get

\[
|B_{n,m}(f)(x) - f(x)| = |f_{j+1,n,j}(x) - f(x)| = f(x) - f_{j+1,n,j}(x) \leq f(x) - f \left( \frac{j}{n} \right)
\]

\[
\leq \omega_1 \left( f; \frac{1}{n} \right)
\]

b) \( B_{n,m}(f)(x) > f(x) \), when \( |B_{n,m}(f)(x) - f(x)| = f_{j+1,n,j}(x) = m_{j+1,n,j}(x)f \left( \frac{j+1}{n} \right) - f(x) \leq f \left( \frac{j+1}{n} \right) - f(x) \). Since \( 0 \leq \frac{j+1}{n} - x = \frac{j+1}{n} - \frac{j}{n} = \frac{1}{n} \leq \frac{1}{n} \), it follows that

\[
f \left( \frac{j+1}{n} \right) - f(x) \leq \omega_1 \left( f; \frac{1}{n} \right)
\]

**Theorem 4.1.**

Let \( f : [0,1] \rightarrow \mathbb{R}_+ \) be continuous. Then we have the estimate

\[
|B_{n,M}(f)(x) - f(x)| \leq 24\omega_1 \left( f; \frac{1}{\sqrt{n+1}} \right), \quad n \in N, \quad x \in [0,1],
\]

where

\[
\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \leq \delta\}.
\]

**Proof.**

One check that the truncated max-product Beta operator execute the condition Corollary 2.3 and we have

\[
|B(f)(x) - f(x)| \leq \left( 1 + \frac{1}{\delta_n} B_{n,M}(\psi_x)(x) \right) \omega_1(f; \delta_n), \quad \ldots (1)
\]

where \( \psi_x(t) = |t - x| \). So, it is enough to estimate

\[
E_n(x) := B_{n,M}(\psi_x)(x) = \frac{\vee_{k=0}^{\sqrt{n}} \beta_{n,k}(x) \left| \frac{k}{n} - x \right|}{\vee_{k=0}^{n} \beta_{n,k}(x)}, \quad x \in [0,1].
\]

Let \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \), where \( j = \{0,1,\ldots,n-1\} \) is fixed. By Lemma 2.4, we obtain

\[
E_n(x) = \max_{k=0,1,2,\ldots} \{M_{k,n,j}(x)\}, \quad x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right],
\]

we suppose that \( j \in \{0,1,\ldots,n-1\} \) because for \( j = 0 \) using evaluation when \( j = 0 \) which implies that, since \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \) then \( x \in \left[ \frac{0}{n}, \frac{0+1}{n} \right] \) implies that \( x \in \left[ \frac{3}{n}, \frac{3}{n} \right] \).
take the value maximum of the interval $\left[0, \frac{1}{n}\right]$ shows that in this case we obtain

$$E_n(x) \leq \frac{1}{n} \text{ for all } x \in \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

Indeed, in this case we get

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left| k - x \right| \left| \frac{k}{n} - x \right| = \frac{(n + k)!}{k! (n - 1)!} \frac{x^k (1 + x)^{-n-k-1}}{j! (n - 1)!} \left| \frac{k}{n} - x \right|$$

and when $j = 0$ implies that $M_{k,n,0}(x) = \frac{\beta_{n,k}(x)}{(1 + x)^{n+1}}$. Indeed, in this case we get

$$M_{0,n,0}(x) = \frac{n!}{(n-1)!} (1 + x)^{-n-1} \left| \frac{k}{n} - x \right| = 1 \cdot x = x.$$  

Therefore, $M_{0,n,0}(x) = x < \frac{1}{n}$, in the interval $\left[0, \frac{1}{n}\right]$. Also, for any $k \geq 1$ we get

$$M_{k,n,0}(x) \leq \frac{\beta_{n,k}(x)}{(1 + x)^{n+1}} \frac{k}{n} \leq \frac{(n + k)!}{n!(k - 1)!} \left( \frac{x}{1 + x} \right)^k \leq \frac{1}{n}.$$  

So, it remain to get an upper estimate for each $M_{k,n,j}(x)$ when $j = \{1, 2, ..., n - 1\}$ is fixed, $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ and $k \in \{0, 1, 2, ..., n\}$. In fact we will prove that $M_{k,n,j}(x) \leq \frac{2\sqrt{3}(\sqrt{2} + 2)}{\sqrt[n+1]{n+1}}$, for all

$$x \in \left[\frac{j}{n}, \frac{j+1}{n}\right], k = \{0, 1, 2, ..., n\} \quad \text{.......... (2)}$$

And which implies that

$$E_n(x) \leq \frac{2\sqrt{3}(\sqrt{2} + 2)}{\sqrt[n+1]{n+1}}, \text{ for all } x \in [0, 1], n \in N, \text{ and taking } \delta_n = \frac{2\sqrt{3}(\sqrt{2} + 2)}{\sqrt[n+1]{n+1}} \text{ in (1), since } [2\sqrt{3}(\sqrt{2} + 2)] = 11, \text{from the property which to say } \omega_1(f; \lambda \delta) \leq (|\lambda| + 1)\omega_1(f; \delta), \text{ we obtain the estimate in the statement. In order to prove (2) we take the following cases: }

1) k = j; \quad 2) k \geq j + 1 \quad \text{and} \quad 3) k \leq j - 1.

Case1). If $k = j$ since

$$M_{k,n,j}(x) = \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)} \left| k - x \right| \text{ then } M_{j,n,j}(x) = \frac{\beta_{n,j}(x)}{\beta_{n,j}(x)} \left| j - x \right| = \left| \frac{j}{n} - x \right| = x - \frac{j}{n},$$

since $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$ then

$$M_{j,n,j}(x) = x - \frac{j}{n} = \frac{j + 1}{n} - \frac{j}{n} = \frac{1}{n}. \text{ It follows that } M_{j,n,j}(x) \leq \frac{1}{n}.$$ 

it follows that $\left| \frac{j+1}{n} - x \right| < \frac{1}{n}$ which implies that $M_{j+1,n,j}(x) < \frac{1}{n}$.

Case2.a) Suppose that $k - \sqrt{2}(k + 1) < j$. We obtain

$$M_{k,n,j}(x) = m_{k,n,j}(x) \left| \frac{k}{n} - x \right| \leq \frac{k}{n} - x \leq \frac{k}{n} - \frac{j}{n} \leq \frac{k}{n} - \frac{k - \sqrt{2}(k + 1)}{n} = \frac{\sqrt{2}(k + 1)}{n} \leq \frac{3\sqrt{2}}{\sqrt[n+1]{n+1}}.$$
Case 2.b). Suppose that \( k - \sqrt{2(k + 1)} \geq j \). Since the function \( g(x) = x - \sqrt{2(x + 1)} \) is nondecreasing on the interval \([0, \infty)\) it follows that there exists \( \bar{k} \in \{0, 1, 2, \ldots, n\} \), of maximum value, such that \( \sqrt{2(\bar{k} + 1)} < j \). Then for \( k_1 = \bar{k} + 1 \) we get \( k_1 - \sqrt{2(k_1 + 1)} \geq j \). Then

\[
M_{\bar{k}+1,n,j}(x) = m_{\bar{k}+1,n,j}(x) \left( \frac{k + 1}{n} - x \right) \leq \frac{k + 1}{n} - x \leq \frac{k + 1}{n} - j
\]

Also, we have \( k_1 \geq j + 1 \). Indeed, this is consequence of the fact the function \( g \) is nondecreasing on the interval \([0, \infty)\) and from simple calculus we obtain \( g(j) < j \). By Lemma 2.3, (i) it follows that \( M_{\bar{k}+1,n,j}(x) \geq M_{\bar{k}+2,n,j}(x) \geq \cdots \geq M_{n,n,j}(x) \).

We hence get on \( M_{k,n,j}(x) \leq \frac{3\sqrt{2} + \sqrt{3}}{\sqrt{n + 1}} \) for any \( k \in \{ \bar{k} + 1, \bar{k} + 2, \ldots, n \} \).

Therefore, in both subcases, by Lemma 3.2, (i), we get \( M_{k,n,j}(x) \leq \frac{3\sqrt{2} + \sqrt{3}}{\sqrt{n + 1}} \).

Case 3.a). In the beginning suppose that \( j - \sqrt{2j} < k \). Then we obtain

\[
M_{k,n,j}(x) = m_{k,n,j}(x) \left( x - \frac{k}{n} \right) \leq \frac{j + 1}{n} - k \leq \frac{j + 1}{n} - j - \sqrt{2j} = \frac{\sqrt{2j} + 1}{n} \leq \frac{\sqrt{2} + 1}{\sqrt{n + 1}}.
\]

b). Suppose now that \( j - \sqrt{2j} \geq k \). Let \( \tilde{k} \in \{0, 1, 2, \ldots, n\} \) be the minimum value such that \( j - \sqrt{2j} < \tilde{k} \). Then \( k_2 = \tilde{k} - 1 \) satisfies \( j - \sqrt{2j} \geq k_2 \). Then

\[
M_{k-1,n,j}(x) = m_{k-1,n,j}(x) \left( x - \frac{k - 1}{n} \right) \leq \frac{j + 1}{n} - \frac{k - 1}{n} \leq \frac{j + 1}{n} - j - \sqrt{2j} = \frac{\sqrt{2j} + 2}{n} \leq \frac{\sqrt{2} + 2}{\sqrt{n}}.
\]

By Lemma 3.4, (ii) it follows that \( M_{k-1,n,j}(x) \geq M_{k-2,n,j}(x) \geq \cdots \geq M_{0,n,j}(x) \).

We get \( M_{k,n,j}(x) \leq \frac{\sqrt{2} + 2}{\sqrt{n}} \) for any \( k \leq j - 1 \) and \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \).

In both subcases, by Lemma 3.2, (ii), we get \( M_{k,n,j}(x) \leq \frac{2(\sqrt{2} + 2)}{\sqrt{n}} \leq \frac{2\sqrt{n} (\sqrt{2} + 2)}{\sqrt{n + 1}} \).

In conclusion, collection all the estimates in the above cases and subcases we obtain the relationship (2), which completes the proof. □

Remark 4.1.

The order of approximation in terms of \( \omega_1 \) in Theorem 4.1 cannot improved (when \( E_n(x) \) is defined in the proof of theorem 4.1 then the expression \( \max_{x \in [0,1]} \{E_n(x)\} \) is exactly \( \frac{1}{\sqrt{n + 1}} \). Indeed, for \( n \in N \) let us take \( j = \left\lfloor \frac{n}{2} \right\rfloor \), \( k_n = j + \left\lfloor \sqrt{n} \right\rfloor \) and \( x_n = \frac{j_n + 1}{n} \). Then using Calculation for all \( n \geq 2 \) we get

\[
M_{k_n,n,j_n}(x_n) = m_{k_n,n,j_n}(x_n) \left( \frac{k_n}{n} - x \right) = \beta_{n,k_n}(x_n) \left( \frac{k_n}{n} - x_n \right) = \frac{(n + k_n)!}{(n + j_n)!} \frac{j_n!}{k_n!} \left( \frac{x_n}{1 + x_n} \right)^{k_n - j_n} \frac{k_n}{n} \left( \frac{k_n}{n} - x_n \right).
\]
Because \( \lim_{n \to \infty} \left( \frac{n + \sqrt{n}}{n + 1} \right)^{\frac{1}{\sqrt{n}}} = e^{-2} \), there exists \( n_0 \in \mathbb{N} \) such that \( \left( \frac{1 + \left[ \frac{n}{2} \right]}{\left[ \sqrt{n} \right]} \right)^{\left[ \sqrt{n} \right]} \geq e^{-3} \), for all \( n \geq \max \{ n_0, 2 \} \). It follows \( M_{k,n,j}(x_n) \geq \frac{e^{-3}}{\sqrt{n+1}}, \) for all \( n \geq \max \{ n_0, 2 \} \).

Taking into Lemma 3.1, (ii) too, It follows that for all \( n \geq \max \{ n_0, 2 \} \) we have

\[
M_{k,n,j}(x_n) \geq \frac{e^{-3}}{\sqrt{n+1}},
\]

which implies the desired conclusion.

**Lemma 4.1.**

Let \( f : [0, 1] \to [0, \infty) \) be concave. Then the function \( g : (0, 1) \to [0, \infty), g(x) = \frac{f(x)}{x} \) is nonincreasing.

**Proof.**

Let \( x, y \in (0, 1) \) be with \( x \leq y \). Then

\[
f(x) = f \left( \frac{x}{y} \right) y + \frac{y - x}{y} 0 \geq \frac{x}{y} f(y) + \frac{y - x}{y} f(0) \geq \frac{x}{y} f(y),
\]

which implies

\[
\frac{f(x)}{x} \geq \frac{f(y)}{y}.
\]

\( \square \)

**Lemma 4.2**

Let \( f : [0, 1] \to [0, \infty) \) be a nondecreasing function such that the function \( g : (0, 1) \to [0, \infty), g(x) = \frac{f(x)}{x} \) is nonincreasing, then

\[
B_{n,M}(f)(x) - f(x) \leq 2\omega_1 \left( \frac{1}{n} \right), \text{ for all } x \in [0, 1], \text{ and } n \in \mathbb{N}.
\]

**Proof.**

Since \( f \) is nondecreasing it follows that

\[
B_{n,M}(f)(x) = \sum_{j \geq \lfloor \frac{j}{n} \rfloor} k \leq \lfloor \frac{j}{n} \rfloor + 1 \]

Let \( x \in [0, 1] \) and \( j \in \{0, 1, 2, \ldots, n - 1\} \) such that \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \). Let \( k \in \{0, 1, \ldots, n\} \) be with \( k \geq j \).
Since
\[ f_{k,n,j}(x) = m_{k,n,j} f \left( \frac{k}{n} \right) = \frac{\beta_{n,k}(x)}{\beta_{n,j}(x)} = \left( \frac{n + k}{n + j} \right) \cdot \frac{j!}{k!} \cdot \left( \frac{x}{1 + x} \right)^{k-j}, \]
\[ f_{k+1,n,j}(x) = m_{k+1,n,j}(x) f \left( \frac{k+1}{n} \right) \]
\[ = \frac{(n + k + 1)!}{(n + j)!} \cdot \frac{j!}{(k + 1)!} \cdot \left( \frac{x}{1 + x} \right)^{k-j} \cdot \left( \frac{x}{1 + x} \right) \cdot f \left( \frac{k + 1}{n} \right). \]
Since \( g(x) \) is nonincreasing we obtain
\[ f \left( \frac{k+1}{n} \right) \leq f \left( \frac{k}{n} \right) \] that is
\[ f \left( \frac{k+1}{n} \right) \leq \frac{k+1}{k} f \left( \frac{k}{n} \right). \]
From \( x \leq \frac{j+1}{n-1} \)
it follows
\[ f_{k+1,n,j}(x) \leq \frac{(n + k + 1)!}{(n + j)!} \cdot \frac{j!}{(k + 1)!} \cdot \left( \frac{x}{1 + x} \right)^{k-j+1} \cdot \frac{n + k}{n + j} \cdot \frac{k + 1}{k} f \left( \frac{k}{n} \right) \]
\[ = f_{k,n,j}(x) \cdot \frac{n + k}{n + j} \cdot \frac{k + 1}{k} \cdot f_{k,n,j}(x) \]
It is directly that for \( k \geq j + 2 \) we have \( f_{k,n,j}(x) \geq f_{k+1,n,j}(x) \). Therefore, we get
\[ f_{j+2,n,j}(x) \geq f_{j+3,n,j}(x) \geq \cdots \geq f_{n,n,j}(x), \]
that is
\[ B_{n,M}(f)(x) = \max \{ f_{j,n,j}(x), f_{j+1,n,j}(x) \}, \]
for all \( x \in \left[ \frac{j}{n}, \frac{j+1}{n} \right] \), and from Lemma 4.1 we get
\[ \left| B_{n,M}(f)(x) - f(x) \right| \leq \omega_1 \left( f; \frac{1}{n} \right). \]

**Corollary 4.3.**

Let \( f : [0,1] \rightarrow [0, \infty) \) be a nondecreasing concave function. Then
\[ \left| B_{n,M}(f)(x) - f(x) \right| \leq \omega_1 \left( f; \frac{1}{n} \right), \]
for all \( x \in [0,1] \).

**Proof.**

The proof immediate by Lemma 4.3 and by Corollary 4.4.

**References**


**Title:** Approximation of truncated Beta operator of Max-product kind

**Abstract:** In this paper, we studied the nonlinear beta operator of the max-product kind and determined its degree of approximation. We also discussed its shape preserving properties.

**Keywords:** Approximation, Beta operator, Max-product kind.